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The random phase variable and quantum optical phase

A Lukš and V Peřinová

Laboratory of Quantum Optics, Faculty of Natural Sciences, Palacký University, Třída Svobody 26, 771 46 Olomouc, Czech Republic

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Abstract. Applying the approach of modelling physical quantities in terms of measurable maps to the phase of a light field, we distinguish a multiple-valued phase from its representation on the unit circle and the single-valued phases in the case of one random phase variable. We study two multiple-valued random phase variables, whose multiple value forms a displaced lattice, and we compare them with their representation on the torus. We relate the phase expectations and variances to single-valued phases, whereas we associate the preferred phase and the phase dispersion with the representation on the circle. We propose alternatives for the covariance of two single-valued random phase variables. Starting from the concept of a characteristic sequence, we introduce the phase characteristics of higher order. As an application to quantum optics, we reformulate the study of phase properties of the states of one- and two-mode light fields.

1. Introduction

The term phase in physics is associated with different meanings; let us compare the physics of phase transitions, the phase space description (representation), and the phase of a process. Classical and quantum theories of a harmonic oscillator and its random phase variable or phase operator are closest to the third notion of phase. In some situations lying beyond the scope of our paper, phases of the Moon are interesting. Speaking of the phases or changes of the Moon, one considers four values of the phase: the New Moon, the First Quarter, the Full Moon, and the Last Quarter.

In the case of the classical harmonic oscillator we can imagine four states of motion, which remind us so much of the phases of the Moon: (i) the oscillating particle at the origin moving to the left; (ii) the left-most position at rest; (iii) the particle at the origin moving to the right; (iv) the right-most position at rest. If we represent the states of motion in the phase space of the harmonic oscillator, the law of energy conservation can be illustrated as the fact that the phase space point moves on a circle and that only a polar angle develops. So the polar angle suffices to describe the phase of motion. It is convenient to represent the phase in the phase space because the position coordinate or deviation of an oscillating particle defines the phase uniquely only at the turning points. Elsewhere, the phase must be made unambiguous regarding the direction of the motion. In optics the complex amplitude of the field belongs to Gauss' plane and if the energy of the monochromatic light field is conserved, the complex amplitude of the field undergoes circular motion just as the classical harmonic oscillator describes. This deep analogue led to the second quantization of electromagnetic fields in terms of a collection of quantum harmonic oscillators. Because in quantum optics the complex amplitude of a Glauber

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coherent state belongs to a kind of phase space [1–4], the coherent state technique came into being and the use of quasidistributions provided evidence of the possibility of a quantum phase. Nevertheless, (quasi)distributions of the quantum phase were not in focus. On the contrary, the cosine- and sine-of-phase operators were considered as the only solution to the quantum phase problem, which indicated simultaneously that similarly built Hermitian harmonic-oscillator phase operators could not be accepted [5, 6]. An advantage of the phase cosine and sine operators is their intimate connection with the phase exponential operators, which act as shift operators on the Fock state representation. Using the cyclic property of shift operators available upon one's restriction to the Hilbert space vectors yielding photon-number distributions of finite variation, we obtain a well behaved Hermitian optical mode phase operator [7]. An immense effort devoted to the quantum phase has been reflected in a special issue of *Physica Scripta* [8]. A thorough discussion of classical, semiclassical, and quantum properties of the phase has been published recently [9].

Even classically the difference between the phase variable and the angle observable is primarily the difference between their conjugate observables. This difference deepens when quantum theory succeeds. In contrast, on neglecting the conjugate variable, there is no difference between the angle and the phase. In this paper we distinguish a multiple-valued random phase variable from single-valued random phase variables and reveal their role in a statistical study and show that this problem becomes more serious when correlation (the statistical dependence between two random phase variables) is interesting. Secondly, we continue with the study of quantum optical models. Similarly, two sections follow, one dealing with one random phase variable and with a pair of random phase variables under the assumption that they are statistically independent, and the other devoted to a study of correlation or statistical dependence between phases. Recognizing the analysis of quantum phase properties of pairs of electromagnetic field modes [10], we approach a study of the parametric down-conversion and we demonstrate ideas and problems connected with the optical phase correlations. We express our orientation to the discussion of other problems related to the phase clearly in section 2, where we combine the examples of classical and quantum distributions and we borrow examples of distributions of quantum origin from the recent discussion [11, 12].

2. One random phase variable and two independent random phase variables

2.1. Theory

The multiple-valued property of phase is obvious. Phase is the polar angle in the phase plane and as such it shares the multivaluedness with the polar angle. To be more definite, let us consider the situation in the descriptive statistics of an experiment, when the measured angle $\overline{\varphi}$ in each trial is recorded like a sequence of points on the real line such that the spacing between any two consecutive points is 2π, $\overline{\varphi} = \{\varphi_0 + 2\pi k; k \in \mathbb{Z}\}$, where $\varphi_0 \in [0, 2\pi)$ and $\mathbb{Z}$ is the set of all integers. After many trials the real line is filled up with records and the distribution may be described by a measure of subsets of the real line. To construct an ideal model of this situation, we note that it is useful to restrict ourselves to subsets, which are, in a sense, 2π-periodic. Paying due attention to the measurable space $(\mathbb{R}, \mathcal{B})$, where $\mathbb{R}$ is the set of real numbers and $\mathcal{B} \equiv \mathcal{B}(\mathbb{R})$ is the σ-field of all Borel subsets of the real line, we introduce the collection of all 2π-periodic Borel sets $B_{2\pi} = \{B \in \mathcal{B}, B = B + 2\pi\}$. This collection is also a σ-field.

To define the multiple-valued random phase variable, we modify the definition of an ordinary single-valued random variable [13, 14]. This modification does not affect the measurable space $(\Omega, \mathcal{F})$, where Ω is the set of elementary random events and $\mathcal{F}$ is a σ-field of random events. The measurable space $(\Omega, \mathcal{F})$ is changed to a probability space $(\Omega, \mathcal{F}, \mathrm{Prob})$ by considering Prob, a probability measure on $\mathcal{F}$.

The most striking modification is the multivaluedness of a random phase variable Φ_{mult} whose domain is Ω and the codomain is $\mathbb{R}$ although this property reduces to a declared equivalence between the propositions: $\varphi_0 = \Phi_{\mathrm{mult}}(\omega)$ with $\varphi_0 \in \mathbb{R}$, $\omega \in \Omega$, and $\varphi_0 + 2\pi k = \Phi_{\mathrm{mult}}(\omega)$ with $k \in \mathbb{Z}$. For any $B \in \mathcal{B}_{2\pi}$ we define $\Phi_{\mathrm{mult}}^{-1}(B) = \{\omega \in \Omega; \Phi_{\mathrm{mult}}(\omega) \in B\}$. Let us note that the property $\Phi_{\mathrm{mult}}(\omega) \in B$ is not contradictory in spite of the multivaluedness, because of $B = B + 2\pi$. We require that $\Phi_{\mathrm{mult}}^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}_{2\pi}$. Then an image of the probability measure on the Borel σ-field is defined such that $\mu_{2\pi}(B) = \mathrm{Prob}\left(\Phi_{\mathrm{mult}}^{-1}(B)\right)$. It may happen, although this is rather a non-constructivist approach, that the image of the probability measure is the Dirac measure $\delta_{2\pi}$ such that $\delta_{2\pi}(B) = 1$ for $B \ni 0$, $\delta_{2\pi}(B) = 0$ otherwise.

Now we leave out the proof of a possibility to extend in a standard way the 'lengths' of the sets $\emptyset$, $\mathbb{R}$, and the sets of the form $I = (a, b) + 2\pi\mathbb{Z}$, where $a < b$, $b - a < 2\pi$. The lengths are as follows: $\nu_{2\pi}(\emptyset) = 0$, $\nu_{2\pi}(R) = 2\pi$, $\nu_{2\pi}(I) = b - a$. If the distribution $\mu_{2\pi}$ is absolutely continuous with respect to the Lebesgue measure $\nu_{2\pi}$, the Radon–Nikodym theorem holds and there exists a probability density $P(\varphi)$ such that

$$\mu_{2\pi}(B) = \int_B P(\varphi)\, \mathrm{d}\nu_{2\pi}(\varphi) \qquad B \in \mathcal{B}_{2\pi}. \tag{2.1}$$

As a consequence of this procedure, the probability density is 2π-periodic,

$$P(\varphi + 2\pi) = P(\varphi) \qquad \varphi \in \mathbb{R}. \tag{2.2}$$

It is a rather different situation from the descriptive statistics of position coordinate x, where one trial provides one recorded point on the x axis and where the obtained density $P(x)$ can be normalized,

$$\int_{-\infty}^{\infty} P(x)\, \mathrm{d}x = 1 \tag{2.3}$$

a contradiction with the property (2.2). Let us consider the formal changes related to the assumption that the phase takes on values in the quotient set $\mathbb{R}/2\pi\mathbb{Z}$, so $\overline{\varphi} \in \mathbb{R}/2\pi\mathbb{Z}$ (they are equivalence classes). Every 2π-periodic function can be treated as defined on this quotient set, first of all $\cos\overline{\varphi}$, $\sin\overline{\varphi}$, with such an argument instead of $\varphi \in \mathbb{R}$ and also the probability density $P(\overline{\varphi})$ can be treated with $\overline{\varphi} \in \mathbb{R}/2\pi\mathbb{Z}$. We will omit the bar over φ in the following. Of course, the equivalence classes φ are residue or congruence classes modulo 2π in this case. The mathematical theory of probability is not too favourable in this respect, because its random variables are measurable mappings of the set Ω of all elementary random events into $\mathbb{R}$. For example, $x = X(\omega)$, where the elementary random event $\omega \in \Omega$. The multivaluedness of phase means a recession from the usual concepts of this theory [15]. Anyway, the random phase variable $\Phi(\omega)$ is rather a measurable mapping of Ω into $\mathbb{R}/2\pi\mathbb{Z}$. In general, 2π-periodic Borel subsets of $\mathbb{R}$ will be treated as Borel subsets of $\mathbb{R}/2\pi\mathbb{Z}$. In this case, there exists a one-to-one correspondence between the measures $\mu_{2\pi}$ and the measures $\overline{\mu}$ on the Borel σ-field $\mathcal{B}(\mathbb{R}/2\pi\mathbb{Z})$. The Dirac measure $\overline{\overline{\delta}}$ is defined so that $\overline{\overline{\delta}}(B) = 1$ for $B \ni 2\pi\mathbb{Z}$, $\overline{\overline{\delta}}(B) = 0$ otherwise. It is illustrative that in this sense the Lebesque measure $\nu_{2\pi}$ becomes the Lebesque measure $\overline{\nu}$ on $\mathcal{B}(\mathbb{R}/2\pi\mathbb{Z})$. The exponential function $\exp(i\varphi) = \cos\varphi + i\sin\varphi$ can be treated as defined for $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, so we may define a pair of random variables $X(\omega)$, $Y(\omega)$ by the equation

$$X(\omega) + \mathrm{i}Y(\omega) = \exp[\mathrm{i}\Phi(\omega)] \qquad \omega \in \Omega. \tag{2.4}$$

With $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, the function $\exp(\mathrm{i}\varphi)$ is an injection (a one-to-one function) into $\mathbb{R}^2$. Its range is the unit circle S^1. Generally, the random vector takes on the values $(X(\omega), Y(\omega)) \in \mathbb{R}^2$, $\omega \in \Omega$. Like this, the theory of probability can be applied again. The measures $\delta_{2\pi}$ and $\bar{\bar{\delta}}$ are replaced by the Dirac formal density $\delta(x-1)\delta(y)$. Let us remark that the replacement of $\Phi(\omega)$ in (2.4) with $Z(\omega)$, a usual random variable, is to be interpreted as wrapping a distribution on a straight line around the unit circle. This wrapping is a reasonable way of conserving the Gaussian distribution for the purpose of the phase. Nevertheless, two random variables sometimes seem to be too many and the circle is mapped into the real line using polar angles in an interval $[\theta_0, \theta_0 + 2\pi)$, where θ_0 is a reference phase. The composite mapping of $\mathbb{R}$ via $\mathbb{R}/2\pi\mathbb{Z}$ via S_1 onto the interval $[\theta_0, \theta_0+2\pi)$ can be plotted as a ratchet-like 2π-periodic function. Conversely, the foregoing statement for $\theta_0 = -\pi$ can be interpreted as a decomposition of the graph presented in [5]. The emerging single-valued random phase variable $\Phi_{\theta_0}(\omega) = \mathrm{Arg}_{\theta_0}\{\exp[\mathrm{i}\Phi(\omega)]\}$, where $\mathrm{Arg}_{\theta_0} z = \mathrm{Im}(\mathrm{Ln}_{\theta_0} z) \in [\theta_0, \theta_0 + 2\pi)$, must have a reference phase. This mapping is by no means unique, in fact, any choice of θ_0 is valid and acceptable, but a choice must be made, e.g., of $\theta_0 = -\pi$ or $\theta_0 = 0$. Here $\mathrm{Ln}_{\theta_0} z$ is the single-valued branch of the natural logarithm defined in the whole complex plane except the ray $z = |z|\exp(\mathrm{i}\theta_0)$ and continued so that on this ray $\mathrm{Ln}_{\theta_0} z = \lim_{z'\to z} \mathrm{Ln}_{\theta_0} z')$ for z' such that $\mathrm{Re}(z'z^*) > 0$, $\mathrm{Im}(z'z^*) \geqslant 0$ hold. The continued example of the Dirac functions is not appropriate here and will be treated below. In the following we will mention also the distributions on the circle [15].

Of course, the above situation of descriptive statistics could have been so approached at once, i.e. with a preselected interval $[\theta_0, \theta_0+2\pi)$, but with respect to the elusive uniqueness of this interval we have started differently. Assuming that the random phase variable $\Phi_{\theta_0}(\omega)$ has a probability density $P_{\theta_0}(\varphi)$, we observe that $P_{\theta_0}(\varphi) = 0$ for φ outside the interval $[\theta_0, \theta_0 + 2\pi)$. The 2π-periodic continuation of this density from $[\theta_0, \theta_0 + 2\pi)$ onto the whole $\mathbb{R}$ can be held for the normalized probability density $P(\varphi)$ defined above. So we may complete that situation with the normalizations

$$\int_{\theta_0}^{\theta_0+2\pi} P(\varphi)\,\mathrm{d}\varphi = 1 \qquad \theta_0 \in \mathbb{R}. \tag{2.5}$$

Vice versa, we observe that $P_{\theta_0}(\varphi) = P(\varphi)$ for $\varphi \in [\theta_0, \theta_0 + 2\pi)$.

It is easy to see that in the study of the phase all information on the phase distribution is contained in a characteristic sequence. This concept is defined by us as

$$\chi(s) = \langle \exp(\mathrm{i}s\Phi)\rangle \qquad s \in \mathbb{Z}. \tag{2.6}$$

Particularly, $\chi(0) = 1$. Assuming the probability density $P(\varphi)$ and choosing $\theta_0 \in \mathbb{R}$, we can express the characteristic sequence as

$$\chi(s) = \int_{\theta_0}^{\theta_0+2\pi} \exp(\mathrm{i}s\varphi) P(\varphi)\,\mathrm{d}\varphi \qquad s \in \mathbb{Z}. \tag{2.7}$$

At each point, where this density is continuous,

$$P(\varphi) = \frac{1}{2\pi}\sum_{s=-\infty}^{\infty} \exp(-\mathrm{i}s\varphi)\chi(s). \tag{2.8}$$

More generally, we denote by $\mathrm{Prob}(E)$ the probability of a random event $E \subset \mathcal{F}$. From the circle topology it is known that an open circular arc Δ is determined by its endpoints θ', θ'' and by some interior point θ. This arc is given by the equation [16]

$$\mathrm{sgn}\, h(\varphi, \theta', \theta'') = \mathrm{sgn}\, h(\theta, \theta', \theta'') \tag{2.9}$$

where

$$h(\varphi, \theta', \theta'') = \tfrac{1}{2}[\sin(\theta' - \varphi) + \sin(\varphi - \theta'') + \sin(\theta'' - \theta')]. \tag{2.10}$$

For an arbitrary Borel set E contained in the unit circle S^1, $E \in \mathcal{B}(S^1)$, we define its measure

$$\mu(E) = \mathrm{Prob}(\{\omega; \exp[\mathrm{i}\Phi(\omega)] \in E\}). \tag{2.11}$$

Assuming $\mu(\{\theta'\}) = \mu(\{\theta''\}) = 0$, we obtain a generalization of (2.8)

$$\frac{\mu(\Delta)}{\nu(\Delta)} = \frac{1}{2\pi} \sum_{s=-\infty}^{\infty} \mathrm{Re}\{\exp(-\mathrm{i}s\theta)\chi(s)\}\mathrm{sinc}[s\nu(\Delta)] \tag{2.12}$$

where $\theta \in \Delta$, $\theta \equiv \frac{1}{2}(\theta' + \theta'')$ modulo 2π, and, on choosing the numbers θ', θ'' to satisfy the inequalities $\theta - \pi \leqslant \theta' < \theta + \pi$, $\theta - \pi \leqslant \theta'' < \theta + \pi$, we obtain that the length of the arc Δ [16],

$$\nu(\Delta) = |\theta'' - \theta'|. \tag{2.13}$$

The function $\mathrm{sinc}\, x$ in (2.12) is defined as

$$\mathrm{sinc}\, x = \begin{cases} \dfrac{\sin(x/2)}{x/2} & \text{for } x \neq 0 \\ 1 & \text{for } x = 0. \end{cases} \tag{2.14}$$

The formula (2.12) can be written equivalently in the form

$$\frac{\mu(\Delta)}{\nu(\Delta)} = \frac{1}{2\pi} \lim_{S\to\infty} \sum_{s=-S}^{S} \exp(-\mathrm{i}s\theta)\chi(s)\mathrm{sinc}[s\nu(\Delta)]. \tag{2.15}$$

Studying an ordinary random variable, we may use the ordinary characteristic function

$$\chi_{\theta_0}(s) = \langle \exp(\mathrm{i}s\Phi_{\theta_0}) \rangle \qquad s \in \mathbb{R}. \tag{2.16}$$

It is connected to the characteristic sequence by the interpolation formula

$$\chi_{\theta_0}(s) = \sum_{s'=-\infty}^{\infty} \chi(s') \exp[\mathrm{i}(s - s')(\theta_0 + \pi)]\mathrm{sinc}[2\pi(s - s')]. \tag{2.17}$$

Vice versa, the characteristic sequence is a restriction of any characteristic function to $\mathbb{Z}$,

$$\chi(s) = \chi_{\theta_0}(s) \qquad s \in \mathbb{Z}. \tag{2.18}$$

Let us recall that the moments of Φ_{θ_0} can be derived from the characteristic function, especially the mean

$$\langle \Phi_{\theta_0} \rangle = \frac{1}{\mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}s} \chi_{\theta_0}(s) \bigg|_{s=0} \tag{2.19}$$

and the second moment

$$\langle \Phi_{\theta_0}^2 \rangle = -\frac{\mathrm{d}^2}{\mathrm{d}s^2} \chi_{\theta_0}(s) \bigg|_{s=0} \tag{2.20}$$

leading to the variance

$$\mathrm{var}(\Phi_{\theta_0}) = \langle (\Delta\Phi_{\theta_0})^2 \rangle \tag{2.21}$$

where

$$\Delta\Phi_{\theta_0} = \Phi_{\theta_0} - \langle \Phi_{\theta_0} \rangle \tag{2.22}$$

so that

$$\mathrm{var}(\Phi_{\theta_0}) = \langle \Phi_{\theta_0}^2 \rangle - \langle \Phi_{\theta_0} \rangle^2. \tag{2.23}$$

In particular, for the Dirac measures $\delta_{2\pi}$, $\bar{\bar{\delta}}$, $\delta(x-1)\delta(y)$ on $(\mathbb{R}, \mathcal{B}_{2\pi})$, $(\mathbb{R}/2\pi\mathbb{Z}, \mathcal{B}(\mathbb{R}/2\pi\mathbb{Z}))$, $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$, respectively, we obtain

$$\langle \Phi_{\theta_0} \rangle = 2\pi \left\lceil \frac{\theta_0}{2\pi} \right\rceil \tag{2.24}$$

where $\lceil x \rceil$ means the least integer greater than or equal to x.

Given the random variable Φ which assumes the values in the quotient set $\mathbb{R}/2\pi\mathbb{Z}$, it is obvious that the random variable $k\Phi$, $k \in \mathbb{Z}$, takes on the values in the quotient set $\mathbb{R}/2\pi k\mathbb{Z}$. Using the natural surjection of $\mathbb{R}/2\pi k\mathbb{Z}$ onto $\mathbb{R}/2\pi\mathbb{Z}$, we get a random variable $\Phi^{(k)}$. According to the definition of the characteristic sequence (2.7) and observing that the random variable $\Phi^{(k)}$ has the density

$$P^{(k)}(\varphi) = \frac{1}{k} \sum_{j=0}^{k-1} P\left(\varphi - \frac{2\pi}{k} j\right) \tag{2.25}$$

we obtain that

$$\chi^{(k)}(s) = \chi(ks) \qquad s \in \mathbb{Z}. \tag{2.26}$$

Using the characteristic sequence, we need a modification of the formula (2.19) for the mean. When $\chi(1) \neq 0$, we define the preferred phase

$$\mathrm{Pref}_{\theta_0} \Phi = \mathrm{Arg}_{\theta_0}[\chi(1)]. \tag{2.27}$$

Recalling the multivaluedness from the analytic function theory, we obtain

$$\mathrm{pref}\, \Phi = \arg[\chi(1)] \tag{2.28}$$

with $\arg z = \mathrm{Im}(\ln z)$. For the Dirac measures, we have

$$\mathrm{Pref}_{\theta_0} \Phi = 2\pi \left\lceil \frac{\theta_0}{2\pi} \right\rceil \tag{2.29}$$

but generally the characteristics (2.27) and (2.19) differ. With the only exception of the random variable being distributed uniformly, there always exists a $\chi(k) \neq 0$ and we define the kth-order preferred phase

$$\mathrm{Pref}_{\theta_0 j}^{[k]} \Phi = \frac{1}{k} \mathrm{Arg}_{k\theta_0}[\chi(k)] + \frac{2\pi}{k} j \qquad j = 0, \ldots, k-1 \tag{2.30}$$

which is not unique but takes on k different values and

$$\mathrm{pref}^{[k]} \Phi = \frac{1}{k} \arg[\chi(k)]. \tag{2.31}$$

These definitions may seem unmotivated but let us remark that there are situations in quantum optics where they can be utilized.

The role of the variance (2.23) is taken over by a less familiar concept of dispersion,

$$D\Phi = 1 - |\chi(1)|^2. \tag{2.32}$$

By analog, we define a kth-order dispersion (cf the case $k = 2$ in [17])

$$D^{[k]}\Phi = \frac{1}{k^2} \left[1 - |\chi(k)|^2\right]. \tag{2.33}$$

In the case of two independent random phase variables Φ_1, Φ_2 with the respective probability densities $P_1(\varphi_1)$, $P_2(\varphi_2)$, the probability densities of their sum and difference

$\Phi_\pm = \Phi_1 \pm \Phi_2$ ($\Phi_\pm(\omega) = \Phi_1(\omega) \pm \Phi_2(\omega)$) in the sense of the quotient set $\mathbb{R}/2\pi\mathbb{Z}$ are given by the convolutions

$$P_\pm(\varphi) = \int_{\theta_0}^{\theta_0+2\pi} P_1(\varphi - \varphi_2) P_2(\pm\varphi_2)\,\mathrm{d}\varphi_2. \tag{2.34}$$

The appropriate dispersions are connected by the relation (see [18] for the phase sum)

$$D(\Phi_\pm) = D(\Phi_1) + D(\Phi_2) - D(\Phi_1)D(\Phi_2). \tag{2.35}$$

A similar relation holds for the second-order dispersions

$$D^{[2]}(\Phi_\pm) = D^{[2]}(\Phi_1) + D^{[2]}(\Phi_2) - 4D^{[2]}(\Phi_1)D^{[2]}(\Phi_2). \tag{2.36}$$

The property of $D(\Phi_1) = 1$ (equivalently $\mathrm{Pref}_{\theta_0}(\Phi_1)$ undefined) implies $D(\Phi_\pm) = 1$ and $\mathrm{Pref}_{\theta_0}(\Phi_\pm)$ undefined.

The familiar theorem of the theory of probability on the characteristic function of the sum of independent random variables has its analogue for the characteristic sequences. It holds that

$$\begin{aligned}\chi_\pm(s) &= \langle \exp(\mathrm{i}s\Phi_\pm)\rangle = \langle \exp(\mathrm{i}s\Phi_1)\exp[\mathrm{i}s(\pm\Phi_2)]\rangle \\ &= \langle \exp(\mathrm{i}s\Phi_1)\rangle\langle \exp[\mathrm{i}s(\pm\Phi_2)]\rangle.\end{aligned} \tag{2.37}$$

According to the definition of the characteristic sequence (2.7) we obtain

$$\chi_+(s) = \chi_1(s)\chi_2(s) \qquad \chi_-(s) = \chi_1(s)\chi_2^*(s) \tag{2.38}$$

where the asterisk denotes the complex conjugation.

A measure of uncertainty in $P(\varphi)$ is provided by the entropy [19]:

$$H = -\int_{\theta_0}^{\theta_0+2\pi} P(\varphi)\ln[P(\varphi)]\,\mathrm{d}\varphi. \tag{2.39}$$

A measure of certainty in $P(\varphi)$ is the Fisher information under the conditions described in [20]:

$$F = \int_{\theta_0}^{\theta_0+2\pi} \left[\frac{\mathrm{d}P(\varphi)}{\mathrm{d}\varphi}\right]^2 \frac{\mathrm{d}\varphi}{P(\varphi)}. \tag{2.40}$$

Here the limits of integration reflect the modification of this concept to the phase. The Cramér–Rao inequality, which assigns the meaning to the Fisher information on the one hand, derives strong uncertainty principles in quantum mechanics and optics on the other hand [21].

The situation in measures of quantum phase uncertainty makes possible or even necessary a comparative study [22,16].

2.2. Application

In applications, we mostly encounter one- and two-peak distributions. We shall first consider two distributions on the circle [15, 23].

(i) For the probability density

$$P(\varphi) = \frac{1}{2\pi I_0(\kappa)} \exp[\kappa\cos(\varphi - \beta)] \qquad \beta \in [\theta_0, \theta_0 + 2\pi) \tag{2.41}$$

the characteristic sequence reads as

$$\chi(s) = \frac{I_s(\kappa)}{I_0(\kappa)} \exp(\mathrm{i}s\beta) \tag{2.42}$$

where $I_s(\kappa)$ is the modified Bessel function of order s. Here

$$\mathrm{Pref}_{\theta_0}\Phi = \beta \tag{2.43}$$

$$D\Phi = 1 - \left[\frac{I_1(\kappa)}{I_0(\kappa)}\right]^2. \tag{2.44}$$

An alternative to the dispersion is provided by the Fisher information

$$F_{\mathrm{I}} = \frac{\kappa^2}{2}\left[1 - \frac{I_2(\kappa)}{I_0(\kappa)}\right]. \tag{2.45}$$

(ii) For the probability density

$$P(\varphi) = \frac{1}{2\pi I_0(\kappa)}\exp\{\kappa\cos[2(\varphi-\beta)]\} \qquad \beta\in[\theta_0,\theta_0+\pi) \tag{2.46}$$

the characteristic sequence reads as

$$\chi(s) = \begin{cases} \dfrac{I_{s/2}(\kappa)}{I_0(\kappa)}\exp(\mathrm{i}s\beta) & \text{for } s \text{ even} \\ 0 & \text{for } s \text{ odd.} \end{cases} \tag{2.47}$$

In this case, $\mathrm{Pref}_{\theta_0}\Phi$ is undefined, $D\Phi = 1$,

$$\mathrm{Pref}^{[2]}_{\theta_0 j}\Phi = \beta + \pi j \qquad j = 0, 1 \tag{2.48}$$

and

$$D^{[2]}\Phi = \frac{1}{4}\left(1 - \left[\frac{I_1(\kappa)}{I_0(\kappa)}\right]^2\right). \tag{2.49}$$

The Fisher information reads

$$F_{\mathrm{II}} = 2\kappa^2\left[1 - \frac{I_2(\kappa)}{I_0(\kappa)}\right]. \tag{2.50}$$

The fact, that the dispersion of any order is a measure of uncertainty and the Fisher information is a measure of certainty can be seen from the proportion

$$\frac{F_{\mathrm{II}}}{F_{\mathrm{I}}} = \frac{D\Phi_{\mathrm{I}}}{D^{[2]}\Phi_{\mathrm{II}}}. \tag{2.51}$$

This comparison would fail if appropriate orders of dispersion were not taken into account.

For quantum motivated phase distributions we take those which played a role in the discussions in [11,12].

(iii) For the probability density

$$P_\beta(\varphi_1) = \frac{1}{2\pi}[1 + \cos(2\varphi_1)] \tag{2.52}$$

the characteristic sequence reads as

$$\chi_\beta(s) = \delta_{s,0} + \tfrac{1}{2}\delta_{s,-2} + \tfrac{1}{2}\delta_{s,2}. \tag{2.53}$$

Here, $\mathrm{Pref}_{\theta_0}(\Phi_1)$ is undefined, $D(\Phi_1) = 1$, and $\mathrm{Pref}^{[2]}_{\theta_0=0,j}(\Phi_1) = \pi j$, $j = 0, 1$, and $D^{[2]}(\Phi_1) = \frac{3}{16}$.

(iv) For the probability density

$$P_\gamma(\varphi_2) = \frac{1}{2\pi}[1 + \cos(\varphi_2 - \theta)] \tag{2.54}$$

the characteristic sequence is

$$\chi_\gamma(s) = \delta_{s,0} + \tfrac{1}{2}\exp(-\mathrm{i}\theta)\delta_{s,-1} + \tfrac{1}{2}\exp(\mathrm{i}\theta)\delta_{s,1}. \tag{2.55}$$

In this case $\mathrm{Pref}_{\theta_0}(\Phi_2) = \theta$, $D(\Phi_2) = \frac{3}{4}$, $\mathrm{Pref}^{[2]}_{\theta_0 j}(\Phi_2)$ are undefined, and $D^{[2]}(\Phi_2) = \frac{1}{4}$.

(v) Pegg and Vaccaro [11] proposed a quantum state $|\gamma\rangle$ and its phase distribution $P_\gamma(\varphi_2)$, related to the random phase variable Φ_2 as being sensitive to the parameter θ. They also considered another quantum state $|\beta\rangle$ and its phase distribution $P_\beta(\varphi_1)$, related to the random phase variable Φ_1. To find the distribution of the phase difference, we observe that $\Phi_1 - \Phi_2 = \Phi_1 + (-\Phi_2)$ and that $-\Phi_2$ has the same distribution as Φ_2 with θ replaced by $-\theta$ ($\theta \to -\theta$). They assumed that the random phase variables Φ_1, Φ_2 were independent. We can see by various arguments that the phase difference is distributed uniformly and does not depend on θ. We use the property that the random phase variable Φ_1 has a symmetry such that it is equal to the random phase variable $\Phi_1 - \pi$ in the distribution, which is π-periodic,

$$P_\beta(\varphi_1 + \pi) = P_\beta(\varphi_1) \tag{2.56}$$

and that the random phase variable $(-\Phi_2)^{(2)}$ is distributed uniformly,

$$P_\gamma(-\varphi_2 - \pi) + P_\gamma(-\varphi_2) = \frac{1}{\pi}. \tag{2.57}$$

Using these properties and the convolution formula (2.34), we obtain that

$$P_-(\varphi) = \frac{1}{2\pi}. \tag{2.58}$$

From relations (2.35) and (2.36), we obtain the dispersion and the second-order dispersion of the phase difference

$$D(\Phi_1 - \Phi_2) = 1 \tag{2.59}$$

and

$$D^{[2]}(\Phi_1 - \Phi_2) = \tfrac{1}{4} \tag{2.60}$$

respectively. These values are appropriate to the uniform distribution, but are not characteristic of it, because the dispersion unity has been obtained in case (iii) and for the second-order dispersion a quarter has been obtained in case (iv).

Substituting (2.53) and (2.55) into (2.38), we obtain that

$$\chi_-(s) = \delta_{s,0}. \tag{2.61}$$

Let us note that this result can be obtained more generally for a pair of independent random phase variables Φ_1, Φ_2 of the properties (2.56) and (2.57), respectively. These properties can be expressed in the language of characteristic sequences as $\chi_\beta(s) = 0$ for s odd and $\chi_\gamma(s) = \delta_{s,0}$ for s even. The random phase variable $(-\Phi_2)^{(2)}$ has the characteristic sequence $\left[\chi_\gamma^{(2)}(s)\right]^* = \chi_\gamma^*(2s) = \delta_{2s,0} = \delta_{s,0}$ in accordance with (2.57).

3. A pair of correlated random phase variables

3.1. Theory

The mathematical theory of probability treats a two-dimensional random vector (X_1, X_2) as a σ-homomorphism of $(\Omega, \mathcal{F})$ into $\left(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)\right)$, where $\mathcal{B}(\mathbb{R}^2)$ is the Borel σ-field on $\mathbb{R}^2$. As was said above, multiple-valued measurable mappings are not usual in this theory and we are inclined to define a two-dimensional random phase vector (Φ_1, Φ_2) as a measurable mapping of $(\Omega, \mathcal{F})$ into $\big((\mathbb{R}/2\pi\mathbb{Z})^2, \mathcal{B}((\mathbb{R}/2\pi\mathbb{Z})^2)\big)$. Fortunately, we may define four random variables $X_1(\omega), Y_1(\omega), X_2(\omega), Y_2(\omega)$ by the equations

$$X_j(\omega) + \mathrm{i}Y_j(\omega) = \exp[\mathrm{i}\Phi_j(\omega)] \qquad j = 1, 2, \omega \in \Omega. \tag{3.1}$$

For $(\varphi_1, \varphi_2) \in (\mathbb{R}/2\pi\mathbb{Z})^2$, the mapping $(\exp(\mathrm{i}\varphi_1), \exp(\mathrm{i}\varphi_2))$ is an injection (a one-to-one function) of $(\mathbb{R}/2\pi\mathbb{Z})^2$ into $\mathbb{C}^2$. Its range is the unit torus T^2. It is worth noting that quite similar equations

$$X_j(\omega) + \mathrm{i}Y_j(\omega) = \exp[\mathrm{i}Z_j(\omega)] \qquad j = 1, 2, \omega \in \Omega \tag{3.2}$$

where $Z_j(\omega)$ are usual random variables, may serve definitions of distributions on the unit torus T^2. To this aim it is sufficient to pick $Z_1(\omega)$, $Z_2(\omega)$ obeying a (normal) Gaussian law. Now, these random variables are formally multiple valued according to their role in relations (3.2). This will lead to summations in the formulae for the new probability densities and simple expressions cannot always be expected from this technique.

The study of two random phase variables seems to necessitate two variables instead of the four and the torus is mapped again into the plane. The simplest approach recently used in the literature on quantum optics [10] consists of considering random phase variables $\Phi_{1\theta_{01}}(\omega)$, $\Phi_{2\theta_{02}}(\omega)$ and their joint probability density distributed on the square $Q = [\theta_{01}, \theta_{01} + 2\pi) \times [\theta_{02}, \theta_{02} + 2\pi)$. Nevertheless, the torus admits various 'charts' even if polar angles are used solely. Some of them are mentioned below, but we adhere to an analysis of the simple chart $\mathrm{Arg}_{\theta_{01}} \times \mathrm{Arg}_{\theta_{02}}$ leading to the relation $\Phi_{j\theta_{0j}}(\omega) = \mathrm{Arg}_{\theta_{0j}}\{\exp[\mathrm{i}\Phi_j(\omega)]\}$, $j = 1, 2$. For simplicity, let us call the set $(2\pi\mathbb{Z})^2$ a lattice and denote it by L. Let us note that $(\mathbb{R}/2\pi\mathbb{Z})^2 = \mathbb{R}^2/L$. Assuming that the random phase variables $\Phi_{1\theta_{01}}(\omega)$, $\Phi_{2\theta_{02}}(\omega)$ have a joint probability density $P_{\theta_{01}\theta_{02}}(\varphi_1, \varphi_2)$, we observe that $P_{\theta_{01}\theta_{02}}(\varphi_1, \varphi_2) = 0$ for (φ_1, φ_2) outside Q. In this paper any doubly 2π-periodic function will be called L-periodic. The L-periodic continuation of the probability density from Q onto the whole $\mathbb{R}^2$ is useful and will be denoted by $P(\varphi_1, \varphi_2)$.

The L-periodicity may motivate the definition of a characteristic double sequence. We introduce this concept as

$$\chi(s_1, s_2) = \langle \exp[\mathrm{i}(s_1\Phi_1 + s_2\Phi_2)] \rangle \qquad (s_1, s_2) \in \mathbb{Z}^2. \tag{3.3}$$

In particular, $\chi(0, 0) = 1$. Supposing that the random phase vector $(\Phi_1(\omega), \Phi_2(\omega))$ has an absolutely continuous distribution with the probability density $P(\varphi_1, \varphi_2)$ and choosing $\theta_{01}, \theta_{02} \in \mathbb{R}$, we can express the characteristic double sequence as

$$\chi(s_1, s_2) = \int_{\theta_{01}}^{\theta_{01}+2\pi} \int_{\theta_{02}}^{\theta_{02}+2\pi} \exp[\mathrm{i}(s_1\varphi_1 + s_2\varphi_2)] P(\varphi_1, \varphi_2) \, \mathrm{d}\varphi_1 \, \mathrm{d}\varphi_2. \tag{3.4}$$

The inverse relation to (3.4) reads

$$P(\varphi_1, \varphi_2) = \frac{1}{4\pi^2} \sum_{s_1=-\infty}^{\infty} \sum_{s_2=-\infty}^{\infty} \exp[-\mathrm{i}(s_1\varphi_1 + s_2\varphi_2)] \chi(s_1, s_2). \tag{3.5}$$

Studying a pair of ordinary random variables, we may use the characteristic function

$$\chi_{\theta_{01}\theta_{02}}(s_1, s_2) = \langle \exp[\mathrm{i}(s_1\Phi_{1\theta_{01}} + s_2\Phi_{2\theta_{02}})] \rangle \qquad (s_1, s_2) \in \mathbb{R}^2. \tag{3.6}$$

It is related to the characteristic double sequence by the interpolation formula

$$\begin{aligned} \chi_{\theta_{01}\theta_{02}}(s_1, s_2) = \sum_{s_1'=-\infty}^{\infty} \sum_{s_2'=-\infty}^{\infty} & \chi(s_1', s_2') \exp\{\mathrm{i}(s_1 - s_1')(\theta_{01} + \pi) + \mathrm{i}(s_2 - s_2')(\theta_{02} + \pi)\} \\ & \times \mathrm{sinc}[2\pi(s_1 - s_1')] \mathrm{sinc}[2\pi(s_2 - s_2')]. \end{aligned} \tag{3.7}$$

The marginal characteristic functions for single random phase variables $\Phi_{1\theta_{01}}$ and $\Phi_{2\theta_{02}}$ are given by the formulae

$$\chi_{\theta_{01}}(s) = \chi_{\theta_{01}\theta_{02}}(s, 0) \qquad \chi_{\theta_{02}}(s) = \chi_{\theta_{01}\theta_{02}}(0, s) \qquad s \in \mathbb{R}. \tag{3.8}$$

Let us remember that the mixed second moment of $\Phi_{1\theta_{01}}$ and $\Phi_{2\theta_{02}}$ is derived as

$$\langle \Phi_{1\theta_{01}} \Phi_{2\theta_{02}} \rangle = -\frac{\partial^2}{\partial s_1 \partial s_2} \chi_{\theta_{01}\theta_{02}}(s_1, s_2)\bigg|_{s_1=0, s_2=0} \tag{3.9}$$

and is used in the computation of the covariance

$$\mathrm{cov}(\Phi_{1\theta_{01}}, \Phi_{2\theta_{02}}) = \langle \Delta\Phi_{1\theta_{01}} \Delta\Phi_{2\theta_{02}} \rangle = \langle \Phi_{1\theta_{01}} \Phi_{2\theta_{02}} \rangle - \langle \Phi_{1\theta_{01}} \rangle \langle \Phi_{2\theta_{02}} \rangle. \tag{3.10}$$

This quantity has been studied in quantum optics [24].

Because the phase-sum and phase-difference variables have been studied intensively [10, 25], we note that the mathematical approach to modelling the physical quantities would require the use of a measurable mapping of $(\Omega, \mathcal{F})$ into $(\mathbb{R}^2/L', \mathcal{B}(\mathbb{R}^2/L'))$ with

$$\begin{aligned} L' &= \{(2\pi(k_1 + k_2), 2\pi(k_1 - k_2)); k_1, k_2 \in \mathbb{Z}\} \\ &= \{(2\pi k_+, 2\pi k_-)\,; k_+, k_- \in 2\mathbb{Z} \text{ or } k_+, k_- \in 2\mathbb{Z} + 1\}\,. \end{aligned} \tag{3.11}$$

Here $2\mathbb{Z} = \{2k; k \in \mathbb{Z}\}$ and $2\mathbb{Z} + 1 = \{2k + 1; k \in \mathbb{Z}\}$. This pair of multiple-valued random variables does not have an L-periodic probability density but an L'-periodic one. This manifests itself in the characteristic double sequence of this pair of multiple-valued random variables, which is given by the formula

$$\chi'_{+-}(s_+, s_-) = \chi(s_+ + s_-, s_+ - s_-) \qquad \text{for } s_+, s_- \in \mathbb{Z} \text{ or } s_+, s_- \in \mathbb{Z} + \tfrac{1}{2} \tag{3.12}$$

where $\mathbb{Z} + \frac{1}{2} = \{k + \frac{1}{2}; k \in \mathbb{Z}\}$, and which provides the L'-periodic probability density as

$$\begin{aligned} P'_{+-}(\varphi_+, \varphi_-) &= \frac{1}{8\pi^2} \sum_{s_+ \in \mathbb{Z}} \sum_{s_- \in \mathbb{Z}} \exp[-\mathrm{i}(s_+\varphi_+ + s_-\varphi_-)]\chi'_{+-}(s_+, s_-) \\ &\quad + \frac{1}{8\pi^2} \sum_{s_+ \in \mathbb{Z}+\frac{1}{2}} \sum_{s_- \in \mathbb{Z}+\frac{1}{2}} \exp[-\mathrm{i}(s_+\varphi_+ + s_-\varphi_-)]\chi'_{+-}(s_+, s_-). \end{aligned} \tag{3.13}$$

Because random phase variables cannot be L'-periodic, previous literature treats the problem of the recovery of the L-periodicity with a procedure. We reapproach this problem via characteristic double sequence. Quite simply

$$\chi_{+-}(s_+, s_-) = \chi'_{+-}(s_+, s_-) \qquad s_+, s_- \in \mathbb{Z}. \tag{3.14}$$

Let us emphasize that the variables s_+, s_- in (3.14) are restricted to the domain of the definition in (3.3).

Returning to formula (3.12), let us remark that the marginal characteristic functions $\chi_+(s)$ and $\chi_-(s)$ for single random phase variables Φ_+ and Φ_-, respectively, can be obtained as usual by substitution $s_- = 0 \in \mathbb{Z}$ and $s_+ = 0 \in \mathbb{Z}$, respectively. So the situation applies with s_+, $s_- \in \mathbb{Z}$ and

$$\chi_+(s) = \chi'_{+-}(s, 0) \qquad \chi_-(s) = \chi'_{+-}(0, s) \qquad s \in \mathbb{Z}. \tag{3.15}$$

By comparison with (3.14) it is obvious that the L'-periodic probability density provides 2π-periodic marginals directly as well as after an intermediate computation of the L-periodic probability density in the formalism of characteristic (double and simple) sequences.

If the random phase variables Φ_1, Φ_2 have a joint probability density $P(\varphi_1, \varphi_2)$, we obtain the L-periodic probability density for the phase sum and phase difference:

$$P_{+-}(\varphi_+, \varphi_-) = \frac{1}{4\pi^2} \sum_{s_+=-\infty}^{\infty} \sum_{s_-=-\infty}^{\infty} \exp[-\mathrm{i}(s_+\varphi_+ + s_-\varphi_-)]\chi_{+-}(s_+, s_-). \tag{3.16}$$

Using (3.4), (3.12), and (3.14), we may rewrite (3.16) as

$$P_{+-}(\varphi_+, \varphi_-) = \iint_Q \bar{\delta}(\varphi_1 + \varphi_2 - \varphi_+)\bar{\delta}(\varphi_1 - \varphi_2 - \varphi_-) P(\varphi_1, \varphi_2)\,\mathrm{d}\varphi_1\,\mathrm{d}\varphi_2 \tag{3.17}$$

where $\bar{\delta}(\varphi)$ is the 2π-periodic Dirac delta function, and $\bar{\delta}(\varphi_1)\bar{\delta}(\varphi_2)$ is the L-periodic Dirac delta function. Using the definition

$$\bar{\delta}(\varphi) = \sum_{k=-\infty}^{\infty} \delta(\varphi - 2\pi k) \tag{3.18}$$

we obtain that the L-periodic probability density restricted to a square Q_{+-} is given by the formulae

$$\begin{aligned} P_{+-}(\varphi_+, \varphi_-) = \frac{1}{2}\bigg[& P_{\theta_{01},\theta_{02}}\left(\frac{\varphi_+ + \varphi_-}{2}, \frac{\varphi_+ - \varphi_-}{2}\right) \\ & + P_{\theta_{01},\theta_{02}}\left(\frac{\varphi_+ + \varphi_-}{2} - \pi, \frac{\varphi_+ - \varphi_-}{2} - \pi\right)\bigg] \end{aligned} \tag{3.19}$$

for $\varphi_+ \geqslant \theta_{0+} + \pi$, $|\varphi_- - \theta_{0-} - \pi| < \varphi_+ - \theta_{0+} - \pi$,

$$\begin{aligned} P_{+-}(\varphi_+, \varphi_-) = \frac{1}{2}\bigg[& P_{\theta_{01},\theta_{02}}\left(\frac{\varphi_+ + \varphi_-}{2}, \frac{\varphi_+ - \varphi_-}{2}\right) \\ & + P_{\theta_{01},\theta_{02}}\left(\frac{\varphi_+ + \varphi_-}{2} - \pi, \frac{\varphi_+ - \varphi_-}{2} + \pi\right)\bigg] \end{aligned} \tag{3.20}$$

for $\varphi_- \geqslant \theta_{0-} + \pi$, $|\varphi_+ - \theta_{0+} - \pi| < \varphi_- - \theta_{0-} - \pi$,

$$\begin{aligned} P_{+-}(\varphi_+, \varphi_-) = \frac{1}{2}\bigg[& P_{\theta_{01},\theta_{02}}\left(\frac{\varphi_+ + \varphi_-}{2}, \frac{\varphi_+ - \varphi_-}{2}\right) \\ & + P_{\theta_{01},\theta_{02}}\left(\frac{\varphi_+ + \varphi_-}{2} + \pi, \frac{\varphi_+ - \varphi_-}{2} + \pi\right)\bigg] \end{aligned} \tag{3.21}$$

for $\varphi_+ \leqslant \theta_{0+} + \pi$, $|\varphi_- - \theta_{0-} - \pi| < |\varphi_+ - \theta_{0+} - \pi|$,

$$\begin{aligned} P_{+-}(\varphi_+, \varphi_-) = \frac{1}{2}\bigg[& P_{\theta_{01},\theta_{02}}\left(\frac{\varphi_+ + \varphi_-}{2}, \frac{\varphi_+ - \varphi_-}{2}\right) \\ & + P_{\theta_{01},\theta_{02}}\left(\frac{\varphi_+ + \varphi_-}{2} + \pi, \frac{\varphi_+ - \varphi_-}{2} - \pi\right)\bigg] \end{aligned} \tag{3.22}$$

for $\varphi_- \leqslant \theta_{0-} + \pi$, $|\varphi_+ - \theta_{0+} - \pi| < |\varphi_- - \theta_{0-} - \pi|$. Here

$$Q_{+-} = [\theta_{0+}, \theta_{0+} + 2\pi) \times [\theta_{0-}, \theta_{0-} + 2\pi) \tag{3.23}$$

with

$$\theta_{0\pm} = (\theta_{01} + \pi) \pm (\theta_{02} + \pi) - \pi = \theta_{01} \pm (\theta_{02} + \pi). \tag{3.24}$$

From (3.19)–(3.22) it is obvious that two points of Q' are cast into a single one of $Q_\pm$.

After a casting (surjection) procedure, the formula $\langle \Phi_{1\theta_{01}} \pm \Phi_{2\theta_{02}} \rangle = \langle \Phi_{1\theta_{01}} \rangle \pm \langle \Phi_{2\theta_{02}} \rangle$ applies no more. A loophole in notation may obscure understanding of a failure of the twin formulae

$$\mathrm{cov}(\Phi_{1\theta_{01}}, \Phi_{2\theta_{02}}) = \pm\tfrac{1}{2}\left[\mathrm{var}(\Phi_{1\theta_{01}} \pm \Phi_{2\theta_{02}}) - \mathrm{var}(\Phi_{1\theta_{01}}) - \mathrm{var}(\Phi_{2\theta_{02}})\right]. \tag{3.25}$$

Recalling the results of Barnett and Pegg [10], we encounter the situation, where the left-hand side of (3.25) depends on a parameter r (see subsection 3.2), but the right-hand side for the phase difference does not depend on r.

The dependence between the random phase variables $\Phi_{1\theta_{01}}$, $\Phi_{2\theta_{02}}$ can be assessed by the normalized covariance (the correlation coefficient)

$$\mathrm{cor}(\Phi_{1\theta_{01}}, \Phi_{2\theta_{02}}) = \frac{\mathrm{cov}(\Phi_{1\theta_{01}}, \Phi_{2\theta_{02}})}{\sqrt{\mathrm{var}(\Phi_{1\theta_{01}})\mathrm{var}(\Phi_{2\theta_{02}})}} \tag{3.26}$$

which can assume values of both signs in the interval $[-1, 1]$. Actually this characteristic measures the dependence between the multiple-valued random phase variables Φ_1, Φ_2, but with a flavour of ambiguity. We assume that a specific concept for the correlation between Φ_1 and Φ_2 should be introduced. To this end we apply the group correlation coefficient [26] between random vectors (X_1, Y_1) and (X_2, Y_2) given in (3.1), using a complexification. This quantity is given as

$$\rho^2_{\exp(\mathrm{i}\Phi_1),\exp(\mathrm{i}\Phi_2)} = 1 - \frac{|\mathbf{W}|}{|\mathbf{V}_{11}||\mathbf{V}_{22}|} \geqslant 0 \tag{3.27}$$

where $|\mathbf{A}|$ means the determinant of a matrix $\mathbf{A}$, and the matrix $\mathbf{W}$

$$\mathbf{W} = \begin{pmatrix} \mathbf{V}_{11} & \mathbf{V}_{12} \\ \mathbf{V}_{21} & \mathbf{V}_{22} \end{pmatrix} \tag{3.28}$$

with

$$\mathbf{V}_{jj} = \begin{pmatrix} \mathrm{cov}(\mathrm{e}^{\mathrm{i}\Phi_j}, \mathrm{e}^{-\mathrm{i}\Phi_j}) & \mathrm{var}(\mathrm{e}^{\mathrm{i}\Phi_j}) \\ \mathrm{var}(\mathrm{e}^{-\mathrm{i}\Phi_j}) & \mathrm{cov}(\mathrm{e}^{\mathrm{i}\Phi_j}, \mathrm{e}^{-\mathrm{i}\Phi_j}) \end{pmatrix} \qquad j = 1, 2 \tag{3.29}$$

$$\mathbf{V}_{12} = \begin{pmatrix} \mathrm{cov}(\mathrm{e}^{\mathrm{i}\Phi_1}, \mathrm{e}^{-\mathrm{i}\Phi_2}) & \mathrm{cov}(\mathrm{e}^{\mathrm{i}\Phi_1}, \mathrm{e}^{\mathrm{i}\Phi_2}) \\ \mathrm{cov}(\mathrm{e}^{-\mathrm{i}\Phi_1}, \mathrm{e}^{-\mathrm{i}\Phi_2}) & \mathrm{cov}(\mathrm{e}^{-\mathrm{i}\Phi_1}, \mathrm{e}^{\mathrm{i}\Phi_2}) \end{pmatrix} \tag{3.30}$$

$$\mathbf{V}_{21} = \mathbf{V}_{12}^{\dagger}. \tag{3.31}$$

The complexification is made without introducing complex conjugation in var ((2.23)) and cov ((3.10)). Motivated by the fact that the coefficient (3.27) cannot take on negative values and by the observation that the diagonal elements on the right-hand side of (3.29)

$$\mathrm{cov}(\mathrm{e}^{\mathrm{i}\Phi_j}, \mathrm{e}^{-\mathrm{i}\Phi_j}) = \langle \Delta \mathrm{e}^{\mathrm{i}\Phi_j} \Delta \mathrm{e}^{-\mathrm{i}\Phi_j} \rangle = D(\Phi_j) \qquad j = 1, 2 \tag{3.32}$$

we introduce the codispersions of the random phase variables Φ_1, Φ_2 and of Φ_1, $-\Phi_2$,

$$\mathrm{cod}(\Phi_1, \Phi_2) = \mathrm{cov}(\mathrm{e}^{\mathrm{i}\Phi_1}, \mathrm{e}^{-\mathrm{i}\Phi_2}) \qquad \mathrm{cod}(\Phi_1, -\Phi_2) = \mathrm{cov}(\mathrm{e}^{\mathrm{i}\Phi_1}, \mathrm{e}^{\mathrm{i}\Phi_2}). \tag{3.33}$$

Let us observe that: (i) we sought for the signs ± 1, but we have obtained complex units because the quantities (3.33) can be imaginary; (ii) we expected only a 'covariance' of Φ_1 and Φ_2, but we have also obtained the codispersion of Φ_1 and $-\Phi_2$. With respect to point (i) we remark that the codispersion can be analysed using the familiar polar decomposition of complex numbers and the squared modulus can be compared with (3.27). As to (ii) we know that the relation between the complex random variables $\exp(\mathrm{i}\Phi_2)$ and $\exp(-\mathrm{i}\Phi_2)$ is nonlinear, i.e. $\exp(-\mathrm{i}\Phi_2) = [\exp(\mathrm{i}\Phi_2)]^{-1}$. In our opinion, no analogue of the twin formulae (3.25) giving a unified measure of correlation in terms of the phase sum and the phase difference is yet known. Exploiting this observation for single phases, we invent also the characteristics

$$\mathrm{cod}(\Phi_j, -\Phi_j) = \mathrm{cov}(\mathrm{e}^{\mathrm{i}\Phi_j}, \mathrm{e}^{\mathrm{i}\Phi_j}) = \mathrm{var}(\mathrm{e}^{\mathrm{i}\Phi_j}) \qquad j = 1, 2. \tag{3.34}$$

To take into account that the random variables (X_1, Y_1), (X_2, Y_2) have the distribution concentrated on the torus T^2, we consider the conditional distribution of the random phase variable Φ_1 given a value φ_2 of the random phase variable Φ_2 and that of Φ_2 given a value φ_1 of Φ_1. Whenever $P(\varphi_2) \neq 0$, $P(\varphi_1) \neq 0$, we introduce the conditional probability densities

$$P(\varphi_1|\varphi_2) = \frac{P(\varphi_1, \varphi_2)}{P(\varphi_2)} \qquad P(\varphi_2|\varphi_1) = \frac{P(\varphi_1, \varphi_2)}{P(\varphi_1)} \tag{3.35}$$

respectively. Applying relation (2.6), we obtain the conditional characteristic sequences $\chi_1(s|\varphi_2)$, $\chi_2(s|\varphi_1)$, but definition (2.5) is modified to the forms

$$\chi_1(s|\varphi_2) = E(\exp(\mathrm{i}s\Phi_1)|\varphi_2) \qquad \chi_2(s|\varphi_1) = E(\exp(\mathrm{i}s\Phi_2)|\varphi_1) \tag{3.36}$$

where E means the conditional expectation,

$$\begin{aligned} E(\exp(\mathrm{i}s\Phi_1)|\varphi_2) &= \int_{\theta_{01}}^{\theta_{01}+2\pi} \exp(\mathrm{i}s\varphi_1) P(\varphi_1|\varphi_2)\,\mathrm{d}\varphi_1 \\ E(\exp(\mathrm{i}s\Phi_2)|\varphi_1) &= \int_{\theta_{02}}^{\theta_{02}+2\pi} \exp(\mathrm{i}s\varphi_2) P(\varphi_2|\varphi_1)\,\mathrm{d}\varphi_2. \end{aligned} \tag{3.37}$$

As a further characteristic of correlation, we may recommend the conditional preferred phases $\mathrm{pref}(\Phi_1|\varphi_2)$, $\mathrm{pref}(\Phi_2|\varphi_1)$ provided that these quantities are defined for all φ_2, φ_1, respectively. With the conditional preferred phases we associated connected lines $l_{1|2}$ and $l_{2|1}$ on the torus T^2 consisting of the points $(\exp[\mathrm{pref}(\Phi_1|\varphi_2)], \exp(\mathrm{i}\varphi_2))$ for all $\varphi_2 \in [\theta_{02}, \theta_{02} + 2\pi)$ and $(\exp(\mathrm{i}\varphi_1), \exp[\mathrm{pref}(\Phi_2|\varphi_1)])$ for all $\varphi_1 \in [\theta_{01}, \theta_{01} + 2\pi)$, respectively. In a situation, to chart the connected line $l_{1|2}$ on the torus T^2, we choose a reference point with the coordinates $\varphi_{1\theta_{01}} = \mathrm{Pref}_{\theta_{01}}(\Phi_1|\varphi_2 = \theta_{02})$, $\varphi_{2\theta_{02}} = \theta_{02}$ in $\mathbb{R}^2$. For $\varphi_{2\theta_{02}} \neq \theta_{02}$, we pick up the values from $\mathrm{pref}(\Phi_1|\varphi_2)$, which will form an image of the line $l_{1|2}$ in $\mathbb{R}^2$ going through the reference point. This procedure is known in optical literature as the phase unwrapping. An image of the line $l_{2|1}$ may be obtained by selecting the reference point with the coordinates $\varphi_{1\theta_{01}} = \theta_{01}$, $\varphi_{2\theta_{02}} = \mathrm{Pref}_{\theta_{02}}(\Phi_2|\varphi_1 = \theta_{01})$ in $\mathbb{R}^2$ and continuing the curve in $\mathbb{R}^2$ with suitable points from $\mathrm{pref}(\Phi_2|\varphi_1)$. The situations of better or worse fit between the curves $l_{1|2}$ and $l_{2|1}$ may be interpreted as a stronger or weaker correlation between the random phase variables Φ_1 and Φ_2. A prospective use can also be associated with the conditional kth-order preferred phases.

Let us derive a decomposition of the covariance of random variables $\exp(\mathrm{i}\Phi_1)$, $\exp(-\mathrm{i}\Phi_1)$. It holds that

$$\begin{aligned} \mathrm{cov}(\mathrm{e}^{\mathrm{i}\Phi_1}, \mathrm{e}^{-\mathrm{i}\Phi_1}) &= \langle 1\rangle - \langle \mathrm{e}^{\mathrm{i}\Phi_1}\rangle\langle \mathrm{e}^{-\mathrm{i}\Phi_1}\rangle = \langle 1\rangle - E_2(E_1(\mathrm{e}^{\mathrm{i}\Phi_1}|\Phi_2)E_1(\mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2)) \\ &\quad + E_2(E_1(\mathrm{e}^{\mathrm{i}\Phi_1}|\Phi_2)E_1(\mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2)) - \langle \mathrm{e}^{\mathrm{i}\Phi_1}\rangle\langle \mathrm{e}^{-\mathrm{i}\Phi_1}\rangle = E_2(E_1(1|\Phi_2) \\ &\quad - E_1(\mathrm{e}^{\mathrm{i}\Phi_1}|\Phi_2)E_1(\mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2)) + E_2(E_1(\mathrm{e}^{\mathrm{i}\Phi_1}|\Phi_2)E_1(\mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2)) \\ &\quad - E_2E_1(\mathrm{e}^{\mathrm{i}\Phi_1}|\Phi_2)E_2E_1(\mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2) = E_2\ \mathrm{cov}_1(\mathrm{e}^{\mathrm{i}\Phi_1}, \mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2) \\ &\quad + \mathrm{cov}_2(E_1(\mathrm{e}^{\mathrm{i}\Phi_1}|\Phi_2), E_1(\mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2)). \end{aligned}$$

We have established that

$$\mathrm{cov}(\mathrm{e}^{\mathrm{i}\Phi_1}, \mathrm{e}^{-\mathrm{i}\Phi_1}) = E_2\ \mathrm{cov}_1(\mathrm{e}^{\mathrm{i}\Phi_1}, \mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2) + \mathrm{cov}_2(E_1(\mathrm{e}^{\mathrm{i}\Phi_1}|\Phi_2), E_1(\mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2)). \tag{3.38}$$

Let us remember that the conditional expectations $E(\mathrm{e}^{\mathrm{i}\Phi_1}|\varphi_2)$, $E(\mathrm{e}^{-\mathrm{i}\Phi_1}|\varphi_2)$ and the conditional covariance $\mathrm{cov}(\mathrm{e}^{\mathrm{i}\Phi_1}, \mathrm{e}^{-\mathrm{i}\Phi_1}|\varphi_2)$ are understandable as related to the conditional probability density $P(\varphi_1|\varphi_2)$ defined in (3.35). Of course, these quantities are functions of the value φ_2 of the random phase variable Φ_2 and when they are measurable functions we may substitute for φ_2 the appropriate random phase variable Φ_2. In this case we use the subscript notation $E_1(\mathrm{e}^{\mathrm{i}\Phi_1}|\Phi_2)$, $E_1(\mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2)$, and $\mathrm{cov}_1(\mathrm{e}^{\mathrm{i}\Phi_1}, \mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2)$ for the random quantities resulting from this substitution. Subscript 2 is used for the expectation of cov_1

and for the expectations and covariances of $E_1(\mathrm{e}^{\mathrm{i}\Phi_1}|\Phi_2)$, $E_1(\mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2)$. A dispersion and the conditional dispersion can be introduced in (3.38) according to formulae (3.32) and to

$$D_1(\Phi_1|\Phi_2) = \mathrm{cov}_1\left(\mathrm{e}^{\mathrm{i}\Phi_1}, \mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2\right). \tag{3.39}$$

Because

$$\begin{aligned}&\mathrm{cov}_2(E_1(\mathrm{e}^{\mathrm{i}\Phi_1}|\Phi_2), E_1(\mathrm{e}^{-\mathrm{i}\Phi_1}|\Phi_2))\\ &\qquad = \mathrm{var}_2\,(E_1(\cos\Phi_1|\Phi_2)) + \mathrm{var}_2(E_1(\sin\Phi_1|\Phi_2)) \geqslant 0\end{aligned} \tag{3.40}$$

it holds that

$$E_2 D_1(\Phi_1|\Phi_2) \leqslant D(\Phi_1). \tag{3.41}$$

The difference (3.40) between the sides of (3.41) is a measure of uncertainty for the prediction of the values of Φ_1 based on the knowledge of Φ_2. Similar considerations are well known for the entropy of the distribution of Φ_1, which enters the right-hand side of an analogue of (3.41), whereas the conditional entropy enters the left-hand side. The right-hand side as reduced by the conditional entropy is called the trans-information [14].

3.2. Application

The technique motivated by the relation (3.2) can be formulated, without introducing $Z_j(\omega)$, in terms of the formula

$$P(\varphi_1, \varphi_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} P_{\mathbb{R}^2}(\varphi_1 - 2\pi k_1, \varphi_2 - 2\pi k_2) \tag{3.42}$$

where $P_{\mathbb{R}^2}(z_1, z_2)$ is a usual probability density, e.g.,

$$\begin{aligned}P_{\mathbb{R}^2}(z_1, z_2) = {}& \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\\ &\times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(z_1-m_1)^2}{\sigma_1^2} + \frac{(z_2-m_2)^2}{\sigma_2^2} - \frac{2\rho(z_1-m_1)(z_2-m_2)}{\sigma_1\sigma_2}\right]\right\}\end{aligned} \tag{3.43}$$

with $\sigma_j > 0$, $j = 1, 2$, $|\rho| < 1$, $m_j \in \mathbb{R}$, $j = 1, 2$. According to a sampling theorem [14] we observe that an equivalent expression consists of (3.5) with

$$\chi(s_1, s_2) = \exp\left[\mathrm{i}(s_1 m_1 + s_2 m_2) - \tfrac{1}{2}(s_1^2\sigma_1^2 + s_2^2\sigma_2^2 + 2s_1 s_2\sigma_1\sigma_2\rho)\right]. \tag{3.44}$$

For $\rho = 0$ the probability density (3.43) is a product of two probability densities and this reappears in (3.42) as the independence of the random phase variables $\Phi_1(\omega)$, $\Phi_2(\omega)$.

The quantum derived phase distribution was considered in an analysis of the two-mode squeezed vacuum state [10]. Here, the L-periodic joint probability density of random phase variables is

$$P(\varphi_1, \varphi_2) = \frac{1}{4\pi^2}\frac{1}{\cosh(2r) - \cos(\varphi_1 + \varphi_2 - \xi)\sinh(2r)} \tag{3.45}$$

where r and ξ are parameters of squeezing, $r \geqslant 0$ and ξ is a phase. The characteristic double sequence (3.3) reads as

$$\chi(s_1, s_2) = \delta_{s_1,s_2}\exp(\mathrm{i}\xi s_1)(\tanh r)^{s_1} \qquad (s_1, s_2) \in \mathbb{Z}^2. \tag{3.46}$$

Considering the L'-periodic probability density of the phase sum and the phase difference Φ_+, Φ_-, we resort to the characteristic double sequence according to (3.12),

$$\chi'_{+-}(s_+, s_-) = \delta_{2s_-,0}\exp(\mathrm{i}\xi s_+)(\tanh r)^{s_+} \qquad \text{for } s_+, s_- \in \mathbb{Z} \text{ or } s_+, s_- \in \mathbb{Z} + \tfrac{1}{2}. \tag{3.47}$$

Adopting the L-periodic probability density, we merely ignore s_+, s_- from $\mathbb{Z}+\frac{1}{2}$ and (3.14) holds. From (3.16) we obtain that

$$P_{+-}(\varphi_+, \varphi_-) = \frac{1}{4\pi^2}\frac{1}{\cosh(2r) - \cos(\varphi_+ - \xi)\sinh(2r)}. \tag{3.48}$$

Trying to apply the specific concept of the group correlation coefficient, we use the matrix (3.28) with

$$\begin{aligned} &\mathbf{V}_{jj} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad j = 1, 2 \\ &\mathbf{V}_{12} = \begin{pmatrix} 0 & e^{i\xi}\tanh r \\ e^{-i\xi}\tanh r & 0 \end{pmatrix} \qquad \mathbf{V}_{21} = \mathbf{V}_{12}^{\dagger}. \end{aligned} \tag{3.49}$$

On substituting into (3.27), we obtain that

$$\rho^2_{\exp(i\Phi_1),\exp(i\Phi_2)} = 1 - \frac{1}{(\cosh r)^4}. \tag{3.50}$$

Analysing the calculations, we may rewrite (3.50) as

$$\rho^2_{\exp(i\Phi_1),\exp(i\Phi_2)} = 1 - [1 - |\mathrm{cod}(\Phi_1, -\Phi_2)|^2]^2. \tag{3.51}$$

Let us note that the marginal phase probability densities are uniform, i.e.

$$P_1(\varphi_1) = P_2(\varphi_2) = \frac{1}{2\pi} \tag{3.52}$$

and the appropriate dispersions are maximum

$$D(\Phi_1) = D(\Phi_2) = 1. \tag{3.53}$$

The codispersions (3.33) are

$$\mathrm{cod}(\Phi_1, \Phi_2) = 0 \qquad \mathrm{cod}(\Phi_1, -\Phi_2) = \exp(i\xi)\tanh r. \tag{3.54}$$

These values have a possible interpretation that the random phase variables Φ_1 and Φ_2 are uncorrelated, but the random phase variables Φ_1 and $-\Phi_2$ are more correlated for greater $\tanh r$ or simply for greater r. The phase parameter ξ could be helpful for choosing the double window Q. According to (3.35) the conditional probability densities are

$$P(\varphi_1|\varphi_2) = P(\varphi_2|\varphi_1) = \frac{1}{2\pi}\frac{1}{\cosh(2r) - \cos(\varphi_1 + \varphi_2 - \xi)\sinh(2r)}. \tag{3.55}$$

The conditional characteristic sequences (3.36) become

$$\chi_1(s|\varphi_2) = \exp[is(\xi - \varphi_2)](\tanh r)^{|s|} \tag{3.56}$$

$$\chi_2(s|\varphi_1) = \exp[is(\xi - \varphi_1)](\tanh r)^{|s|}. \tag{3.57}$$

The conditional preferred phases follow from (3.54) and (3.55),

$$\xi - \varphi_2 \in \mathrm{pref}(\Phi_1|\varphi_2) \qquad \xi - \varphi_1 \in \mathrm{pref}(\Phi_2|\varphi_1). \tag{3.58}$$

Here the graphs of these dependences are determined as

$$\begin{aligned} l_{1|2} &= \{(e^{i(\xi-\varphi_2)}, e^{i\varphi_2}); \varphi_2 \in [\theta_{02}, \theta_{02} + 2\pi)\} \\ l_{2|1} &= \{(e^{i\varphi_1}, e^{i(\xi-\varphi_1)}); \varphi_1 \in [\theta_{01}, \theta_{01} + 2\pi)\} \end{aligned} \tag{3.59}$$

providing the same curve (a helix on a torus, which for a specific closedness of the torus is a topological circle). This identity does not ensure the minimum conditional dispersions,

$$D(\Phi_1|\varphi_2) = D(\Phi_2|\varphi_1) = \frac{1}{\cosh^2 r}. \tag{3.60}$$

Hence,

$$E_2 D_1(\Phi_1|\Phi_2) = E_1 D_2(\Phi_2|\Phi_1) = \frac{1}{\cosh^2 r} \tag{3.61}$$

and in relation (3.41) the equality is not attained.

4. Conclusion

In this paper we have treated the phase as a multiple-valued random variable with and without the formal device of collecting together all assumed values into a single set. We have illustrated the consequences of representing these phase values as points on unit circle. The ordinary treatment of the phase in a suitable interval has been interpreted as a mapping of the unit circle into the real line. The expectation and the variance of the ordinary phase has been exposed together with the preferred phase and the dispersion connected to the unit circle. The characteristic sequence of the random phase variable, which comprises only 'resonant' values from the characteristic functions for the ordinary phase, has been introduced. Starting from this concept, we have defined more general concept of the kth-order preferred phase and the appropriate dispersion. A possible use in the quantum and semiclassical optics is conceived. The same conceptual scheme has been adopted in the case of a pair of random phase variables leading to a representation on the unit torus in four-dimensional space. Here the gap between the dimensionality two of the torus and that of the Euclidean space gets wider and proposals for measures of stochastic dependence (statistical correlation) encounter difficulties. Nevertheless, a choice of a suitable mapping in this two-dimensional case is also a challenge. In this situation we have adapted the group correlation coefficient for the analysis of phase properties. We have reapproached the problem of the phase sum and phase difference and characterized exactly the peculiar properties of this pair of multiple-valued random variables, which arise in the course of the transformation. We have translated these distinctions between lattices into the language of characteristic double sequences and found a common basis for 'casting' procedures known from the literature in the deliberate neglect of the terms of the double sequence indexed with half-odd subscripts. We have presented an explicit form of a possible imposition of the product multivaluedness to the phase sum and the phase difference. As possible measures of correlation, the codispersions of random phase variables have been introduced possessing unusual properties. The codispersions may be imaginary and their definitions respect the phase conjugation. We have pointed out the usefulness of the conditional phase distributions for expressing the dependences once the conditional preferred phases are plotted on the surface of the torus and the averaged conditional dispersions are assessed. We have mentioned an information-based measure. As an application, we have considered the situation, where two-mode squeezed vacuum quantum phases are obtained using an ideal down-converter. Building on the results of Barnett and Pegg, we have illustrated our proposals and concepts.

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Appendix A. Physical quantities: random variables and operators

Modern physics is based on quantum theory. Although our paper is intended as a contribution to a very special problem of modern physics, it hardly contains concepts such as states and operators, but it indicates the use of some distributions of ‘quantum origin’.

Let us begin with the concept of a random variable, which seems to be very mathematical, but it may characterize a physical quantity. For instance, with a position coordinate, the notation $\langle X \rangle$ may be connected with a classical model and $\langle \hat{x} \rangle$ may occur in a quantum model of reality. The expectation value and moments of a random variable $X(\omega)$ are defined as

$$\langle X^k \rangle = \int_{\Omega} [X(\omega)]^k \, \mathrm{d} \, \mathrm{Prob}(\omega) \tag{A1}$$

where Prob is a probability measure on a σ-field $\mathcal{F}$ of subsets of Ω, and Ω is the set of elementary random events. From the physical viewpoint the formula (A1) is too much connected with the Kolmogorov axioms of the probability theory although a simple substitution $x = X(\omega)$ leads to an accepted formula

$$\langle X^k \rangle = \int_{\mathbb{R}} x^k \, \mathrm{d}\mu_X(x) \tag{A2}$$

where the Borel measure

$$\mu_X(E) = \mathrm{Prob}(X^{-1}(E)) \tag{A3}$$

with $E \in \mathcal{B}(\mathbb{R})$. The variance of the random variable $X(\omega)$ is

$$\mathrm{var}\ X = \langle (\Delta X)^2 \rangle \tag{A4}$$

where $\Delta X = X - \langle X \rangle$, and it holds that

$$\mathrm{var}\ X = \langle X^2 \rangle - \langle X \rangle^2. \tag{A5}$$

Modern physics does not adopt the term ‘random variable’, but it still associates the statistical notions with physical quantities, which are now being represented by operators. In non-commutative measure theory, which is being developed because of the desire to investigate the mathematical foundations of quantum mechanics (see [27]), one replaces the notion of a Boolean algebra by the notion of an orthomodular lattice. In contrast, here we use the representation of a Boolean algebraic structure with a σ-field of sets as usual in probability theory. In the classical model the position coordinate is the random variable $X(\omega)$ and in quantum optics the position coordinate is represented by the operator $\hat{x}$. We will denote operators using the caret. The expectation value and moments of this operator are defined as

$$\langle \hat{x}^k \rangle = \mathrm{Tr}\{\hat{\rho}\hat{x}^k\} \tag{A6}$$

where $\hat{\rho}$ is the state operator. In the quantum theory of measurement, a measure on the Borel σ-field $\mathcal{B}(\mathbb{R})$ is introduced and by analogue with (A3) we denote it similarly,

$$\mu_{\hat{x}}(E) = \mathrm{Tr}\{\hat{\rho}\hat{\Delta}_{\hat{x}}(E)\} \tag{A7}$$

where $\hat{\Delta}_{\hat{x}}(E)$ is a projection-valued measure with the property that $\hat{\Delta}_{\hat{x}}(\mathbb{R}) = \hat{1}$. Upon substituting $\mu_{\hat{x}}$ for μ_X into (A2) and using the spectral decompositions

$$\int_{\mathbb{R}} x^k \hat{\Delta}_{\hat{x}}(\mathrm{d}x) = \hat{x}^k \tag{A8}$$

and the definition (A6), we obtain the relation

$$\langle X^k \rangle = \langle \hat{x}^k \rangle. \tag{A9}$$

This is why we have generalized the use of the left-hand side of (A6) and that of random variables to the whole of this paper. The variance of the operator $\hat{x}$ is

$$\operatorname{var} \hat{x} = \langle (\Delta\hat{x})^2 \rangle \tag{A10}$$

where $\Delta\hat{x} = \hat{x} - \langle \hat{x} \rangle \hat{1}$, and equivalently

$$\operatorname{var} \hat{x} = \langle \hat{x}^2 \rangle - \langle \hat{x} \rangle^2. \tag{A11}$$

A similar replacement of the random variables by operators can be performed in the connection with the notation $\operatorname{cov}(X_1, X_2)$ for the covariance and $E(X_1|x_2)$, $E_1(X_1|X_2)$ for the expectation values, which in the latter case itself is a random variable, i.e. $\operatorname{cov}(\hat{x}_1, \hat{x}_2)$, $E(\hat{x}_1|x_2)$, and $E_1(\hat{x}_1|\hat{x}_2)$. The expectation value $E_1(\hat{x}_1|\hat{x}_2)$ is defined only in case that the operators $\hat{x}_1, \hat{x}_2$ commute, i.e. the physical quantities are compatible, and then $E_1(\hat{x}_1|\hat{x}_2)$ is an operator.

The preceding scheme of reinterpretation of random variables as operators can be applied in simple cases, e.g., the left-hand sides of the formulae (2.19)–(2.21), if we adopt a suitable phase identity resolution. Unfortunately, the Hilbert space of the harmonic oscillator does not admit a well behaved phase identity resolution and there exist two contrasted solutions of the phase operator problem when this Hilbert space is enlarged [28] or diminished [7, 29].

We may start the procedure of enlargement with a continuum of rotation angle states $|\varphi\rangle_{\mathrm{e}}, \varphi \in \mathbb{R}/2\pi\mathbb{Z}$, of the axial rotator. Formally a simple Fourier decomposition yields number states,

$$|n\rangle_{\mathrm{e}} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}/2\pi\mathbb{Z}} \exp(-\mathrm{i}n\varphi)|\varphi\rangle_{\mathrm{e}}\,\mathrm{d}\varphi \qquad n \in \mathbb{Z}. \tag{A12}$$

When we identify the number states $|n\rangle_{\mathrm{e}}$, $n \geqslant 0$, with number states $|n\rangle$ of the harmonic oscillator, we reinterpret the states $|\varphi\rangle_{\mathrm{e}}$ as phase states of the harmonic oscillator. We may associate any study of the physical system with states, which do not comprise components with negative energy of the harmonic oscillator. However, these components enter the expansion of the phase states, which may represent a problem for their physical acceptance. Whereas the distinction between the probability densities $P_{\theta_0}(\varphi)$ (cf (2.5)) is concrete, the distinction between measures $\mu_{\varphi_{\theta_0}}(E)$, $E \in \mathcal{B}(\mathbb{R})$, is abstract. In analog to (A7), a similar family of measures of quantum origin can be obtained,

$$\mu_{\hat{\varphi}_{\theta_0}}(E) = \operatorname{Tr}\{\hat{\rho}\hat{\Delta}_{\hat{\varphi}_{\theta_0}}(E)\} \tag{A13}$$

where $\hat{\Delta}_{\hat{\varphi}_{\theta_0}}(E)$ is in fact independent of θ_0. This property can be illustrated by the operator density $\mathrm{d}/\mathrm{d}\varphi\hat{\Delta}_{\hat{\varphi}_{\theta_0}}(\varphi)$ which vanishes outside the interval $[\theta_0, \theta_0 + 2\pi)$, but whenever it is non-zero at φ_1, φ_2 (in other words, for $\theta_0 = \theta_{01}, \theta_0 = \theta_{02}$, respectively, $\varphi_j \in [\theta_{0j}, \theta_{0j} + 2\pi)$, $j = 1, 2$), and $\varphi_1 \equiv \varphi_2$ modulo 2π, the following relation holds:

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\hat{\Delta}_{\hat{\varphi}_{\theta_{01}}}(\varphi)|_{\varphi=\varphi_1} = \frac{\mathrm{d}}{\mathrm{d}\varphi}\hat{\Delta}_{\hat{\varphi}_{\theta_{02}}}(\varphi)|_{\varphi=\varphi_2}. \tag{A14}$$

Let us remember the formula (2.27) in the form

$$\operatorname{Pref}_{\theta_0}\Phi = \operatorname{Arg}_{\theta_0}\langle \exp(\mathrm{i}\Phi) \rangle. \tag{A15}$$

The expectation value on the right-hand side in (A15) is defined as

$$\langle \exp(\mathrm{i}\Phi) \rangle = \int_{\Omega} \exp[\mathrm{i}\Phi(\omega)]\,\mathrm{d}\,\mathrm{Prob}(\omega). \tag{A16}$$

With the aid of the substitution $\varphi = \Phi(\omega)$, which is not so usual, because $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$, we arrive at

$$\langle \exp(\mathrm{i}\Phi)\rangle = \int_{\mathbb{R}/2\pi\mathbb{Z}} \exp(\mathrm{i}\varphi)\,\mathrm{d}\overline{\mu}(\varphi). \tag{A17}$$

Let us rewrite (2.32) in the form

$$\mathrm{D}\Phi = 1 - |\langle \exp(\mathrm{i}\Phi)\rangle|^2. \tag{A18}$$

Although the notations Pref and D have been introduced in this paper, we would like to use the same principle as with the usual notation and replace the random phase variable with an operator $\hat{\varphi}$. This means, that using the comfort of the enlarged Hilbert space, we want to do away with the subscript θ_0. We will show that this is not possible. Only supposing for a while the existence of the multiple-valued operator, we may rewrite formally (A15) as

$$\mathrm{Pref}_{\theta_0}\hat{\varphi} = \mathrm{Arg}_{\theta_0}\langle \exp(\mathrm{i}\hat{\varphi})\rangle. \tag{A19}$$

This situation does not oblige us to complete the definition of the operator $\hat{\varphi}$ at once because obviously

$$\exp(\mathrm{i}\hat{\varphi}) = \widehat{\exp}(\mathrm{i}\varphi) \tag{A20}$$

where $\widehat{\exp}(\mathrm{i}\varphi)$ is a ladder operator in the enlarged Hilbert space, and

$$\langle \widehat{\exp}(\mathrm{i}\varphi)\rangle = \mathrm{Tr}\{\hat{\rho}\ \widehat{\exp}(\mathrm{i}\varphi)\}. \tag{A21}$$

Before we approach the possibility of and constraints on the multiple-valued phase operator, we remember that the following equality between the multiple-valued random phase variables holds:

$$\Phi_{\mathrm{mult}}(\omega) = \Phi_{\theta_0}(\omega) + 2\pi k \qquad k \in \mathbb{Z}. \tag{A22}$$

In other words, the addition of an indeterminate multiple of 2π to any 'well behaved' random phase variable provides just the multiple-valued random phase variable. As the matrix elements of the multiple-valued phase operator

$$\hat{\varphi}_{\theta_0 \mathrm{mult}} = \hat{\varphi}_{\theta_0} + 2\pi k \hat{1}_{\mathrm{e}} \tag{A23}$$

where $\hat{1}_{\mathrm{e}}$ is the identity operator in the enlarged Hilbert space, depend on θ_0 modulo 2π, relation (A23) fails to define the operator $\hat{\varphi}_{\mathrm{mult}}$. Nevertheless, we may imagine a multiple-valued measurement with the numerical result $\varphi \in \mathbb{R}/2\pi\mathbb{Z}$ independent of θ_0 and an immediately following state reduction to the pure state $|\varphi\rangle_{\mathrm{ee}}\langle\varphi|$, again independent of θ_0. Also the appropriate projection-valued measure $\hat{\Delta}_{\hat{\varphi}}(E)$, $E \in \mathcal{B}(\mathbb{R}/2\pi\mathbb{Z})$, may be provided with a subscript $\hat{\varphi}$. Applying the previous convention, we generalize relations (3.33) as

$$\mathrm{cod}(\hat{\varphi}_1, \pm\hat{\varphi}_2) = \mathrm{cov}(\exp(\mathrm{i}\hat{\varphi}_1), \exp(\mp\mathrm{i}\hat{\varphi}_2)). \tag{A24}$$

Similarly, formula (3.58) can be rewritten with $\mathrm{pref}(\hat{\varphi}_1|\varphi_2)$, $\mathrm{pref}(\hat{\varphi}_2|\varphi_1)$.

References

[1] Glauber R J 1963 *Phys. Rev. Lett.* **10** 84
[2] Glauber R J 1963 *Phys. Rev.* **130** 2529
[3] Glauber R J 1963 *Phys. Rev.* **131** 2766
[4] Sudarshan E G C 1963 *Phys. Rev. Lett.* **10** 277
[5] Susskind L and Glogower J 1964 *Physics* **1** 49
[6] Carruthers P and Nieto M 1968 *Rev. Mod. Phys.* **40** 411
[7] Pegg D T and Barnett S M 1988 *Europhys. Lett.* **6** 483

[8] 1993 *Phys. Scr.* (Special issue) **T48** 1–144
[9] Royer A 1996 *Phys. Rev.* A **53** 70
[10] Barnett S M and Pegg D T 1990 *Phys. Rev.* A **42** 6713
[11] Pegg D T and Vaccaro J A 1995 *Phys. Rev.* A **51** 859
[12] Luis A and Sánchez-Soto L L 1995 *Phys. Rev.* A **51** 861
[13] Kolmogoroff A 1950 *Foundations of the Theory of Probability* (New York: Chelsea)
[14] Frieden B R 1991 *Probability, Statistical Optics, and Data Testing* 2nd edn (Berlin: Springer)
[15] Rao R C 1973 *Linear Statistical Inference and its Applications* (New York: Wiley) p 175
[16] Lukš A and Peřinová V 1994 *Quantum Opt.* **6** 125
[17] Vaccaro J A and Orlowski A 1995 *Phys. Rev.* A **51** 4172
[18] Opatrný T 1994 *J. Phys. A: Math. Gen.* **27** 7201
[19] Bialynicki-Birula I and Mycielski J 1975 *Commun. Math. Phys.* **44** 129
[20] Frieden B R 1990 *Phys. Rev.* A **41** 4265
[21] Frieden B R 1992 *Phys. Lett.* A **169** 123
[22] Bialynicki-Birula I, Freyberger M and Schleich W 1993 *Phys. Scr.* **T48** 113
[23] Lukš A and Peřinová V 1991 *Czech. J. Phys.* **41** 1205
[24] Tanaś R, Gantsog Ts and Zawodny R 1991 *Quantum Opt.* **3** 221
[25] Luis A and Sánchez-Soto L L 1993 *Phys. Rev.* A **48** 4702
[26] Anděl J 1978 *Mathematical Statistics* (Prague: Technical Literature Publishers) (in Czech) p 307
[27] Varadarajan V S 1985 *Geometry of Quantum Theory* (Berlin: Springer)
[28] Lukš A, Peřinová V and Křepelka J 1994 *Phys. Rev.* A **50** 818
[29] Lukš A and Peřinová V 1993 *Quantum Opt.* **5** 287